MULTIDIMENSIONAL CONTINUED FRACTIONS AND A MINKOWSKI FUNCTION

GIOVANNI PANTI

Abstract. The Minkowski Question Mark function can be characterized as the unique homeomorphism of the real unit interval that conjugates the Farey map with the tent map. We construct an n-dimensional analogue of the Minkowski function as the only homeomorphism of an n-simplex that conjugates the piecewise-fractional map associated to the Mönkemeyer continued fraction algorithm with an appropriate tent map.

1. Preliminaries

The nth order *Farey set* $\mathcal{F}_n$ in the real unit interval $[0,1]$ is defined by recursion: one starts with $\mathcal{F}_0 = \{0/1, 1/1\}$ and obtains $\mathcal{F}_n$ by adding to $\mathcal{F}_{n-1}$ all the *Farey sums* $v_1 \oplus v_2 = (a_1+a_2)/(b_1+b_2)$ of two consecutive elements $v_i = a_i/b_i$ of $\mathcal{F}_{n-1}$. The union of all the $\mathcal{F}_n$'s is the set of all rational numbers in $[0,1]$. Analogously, by starting with $\mathcal{B}_0 = \mathcal{F}_0$ and replacing the Farey sum with the *barycentric sum* $v_1 \boxplus v_2 = (v_1+v_2)/2$, we obtain an increasing sequence $\mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{B}_2 \subset \cdots$, whose union is the set of all dyadic rationals in $[0,1]$. For every $n \geq 0$, there exists a unique order-preserving bijection from $\mathcal{F}_n$ to $\mathcal{B}_n$. The union of these bijections is a bijection from $\bigcup_{n\geq 0} \mathcal{F}_n$ to $\bigcup_{n\geq 0} \mathcal{B}_n$, which extends uniquely by continuity to an order-preserving bijection $\Phi : [0,1] \to [0,1]$. This last map is the Minkowski Question Mark function [7], [15], [13], [20]. Among others, Φ has the following properties:

(1) it is an order-preserving homeomorphism of $[0,1]$;
(2) it maps bijectively the rational numbers to the dyadic rationals, and the real algebraic numbers of degree ≤ 2 to the rationals (all these sets restricted to $[0,1]$, of course);
(3) it is singular w.r.t. the Lebesgue measure λ (i.e., there exists a measurable set $A \subseteq [0,1]$ such that $\lambda(A) = 1$ and $\lambda\big(\Phi[A]\big) = 0$);
(4) it has a fractal structure —which is apparent in the following approximate sketch—

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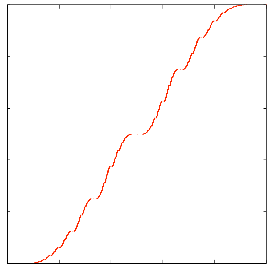


(5) it conjugates the *Farey map*

$$F(x) = \begin{cases} x/(1-x), & \text{if } 0 \le x < 1/2; \\ (1-x)/x, & \text{if } 1/2 \le x \le 1; \end{cases}$$

with the *tent map*

$$T(x) = \begin{cases} 2x, & \text{if } 0 \le x < 1/2; \\ 2-2x, & \text{if } 1/2 \le x \le 1. \end{cases}$$

Property (4) means, more precisely, the following: let $v_1 < v_2$ be consecutive elements of some $\mathcal{F}_n$. Then there exists a unique element $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of $\mathrm{PSL}_2\,\mathbb{Z}$ such that the corresponding fractional-linear transformation $G(x) = (ax+b)/(cx+d)$ maps v_1 to 0 and v_2 to 1. Analogously, there is a unique affine transformation $H(x) = rx + s$ such that $H(\Phi(v_1)) = 0$ and $H(\Phi(v_2)) = 1$. One then checks easily that $\Phi = H \circ (\Phi \restriction [v_1, v_2]) \circ G^{-1}$.

We note the following for future reference.

Proposition 1.1. *Property (5) characterizes* Φ.

Proof. Let Ψ be a homeomorphism of $[0,1]$ such that $T = \Psi \circ F \circ \Psi^{-1}$. The only point which is fixed by F (respectively, T), and whose removal does not disconnect $[0,1]$ is 0; therefore $\Psi(0) = 0$ and Ψ is order-preserving. For every $n \ge 0$ we have $\mathcal{F}_n = F^{-(n+1)}\{0\}$ and $\mathcal{B}_n = T^{-(n+1)}\{0\}$. Hence, for every n, Ψ restricts to a bijection between $\mathcal{F}_n$ and $\mathcal{B}_n$. Since these bijections are order-preserving, Ψ must coincide with Φ. $\square$

In [2] a generalization of the Minkowski function to a selfmap δ of a 2-dimensional triangle is proposed. The construction of δ proceeds in stages, and parallels that for Φ: assume that $\langle v_1^1, v_2^1, v_3^1 \rangle$ and $\langle v_1^2, v_2^2, v_3^2 \rangle$ are paired triangles that appear at the $(n-1)$th stage of the construction "on the Farey side" and "on the barycentric side", respectively. Then, at the nth stage, $\langle v_1^i, v_2^i, v_3^i \rangle$ is subdivided into three subtriangles $\langle v_1^i, v_2^i, w^i \rangle$, $\langle v_1^i, w^i, v_3^i \rangle$, $\langle w^i, v_2^i, v_3^i \rangle$, where w^1 is the Farey sum of v_1^1, v_2^1, v_3^1, and w^2 is the barycentric sum of v_1^2, v_2^2, v_3^2. The new triangles are paired in the obvious way, and the function δ is defined using an appropriate limiting process. This δ function is not injective, nor is continuous at all points [2, p. 117]. This is essentially due to the fact that not every sequence of nested Farey triangles $\langle v_1^1(0), v_2^1(0), v_3^1(0) \rangle \supset \langle v_1^1(1), v_2^1(1), v_3^1(1) \rangle \supset \langle v_1^1(2), v_2^1(2), v_3^1(2) \rangle \supset \cdots$ intersects in a single point (here, for every n, $\langle v_1^1(n), v_2^1(n), v_3^1(n) \rangle$ is one of the three triangles resulting from the subdivision of $\langle v_1^1(n-1), v_2^1(n-1), v_3^1(n-1) \rangle$ at stage n). In terms of multidimensional continued fractions [4] [17], this amounts to saying that

the continued fraction algorithm naturally associated with the 2-dimensional Farey partition is not topologically convergent [17, Definition 9].

In this paper we will construct another generalization of the Minkowski function, by replacing the 2-dimensional Farey continued fraction algorithm with the n-dimensional Mönkemeyer algorithm. The latter algorithm is topologically convergent, and this fact allows us to construct, for every $n \geq 2$, an n-dimensional Minkowski function Φ which is a homeomorphism. We will show that appropriate analogs of the properties (1)–(5) continue to hold, with the exception of (2), for which we have partial results.

A remark on terminology is in order here: the multidimensional continued fraction algorithm we are going to use has been rediscovered over and over again. We call it the *Mönkemeyer algorithm* —and we call *Mönkemeyer map* the associated piecewise-fractional map— since the first reference we are aware of is [14]. The name *Selmer algorithm* is more widely used; as a matter of fact, the Mönkemeyer algorithm is just the restriction of the Selmer one [18] to the absorbing simplex D of [17, Theorem 22]. In [1] the same algorithm is called the GMA (generalized mediant algorithm), and is defined on a simplex obtainable from D via a permutation of the coordinates. All these versions are easily shown to be equivalent to each other.

2. An n-dimensional Minkowski function

We will define our generalization Φ of the Minkowski function as the only homeomorphism of a certain n-dimensional simplex Δ that conjugates the Mönkemeyer map M with a version of the tent map T, both maps to be defined shortly. In order to streamline the presentation, we fix some notation. First of all, we fix an integer $n \geq 1$, and we identify $\mathbb{R}^n$ with the plane $\pi = \{x_{n+1} = 1\}$ in $\mathbb{R}^{n+1}$. If $\mathbf{v} = (\alpha_1, \dots, \alpha_n, \alpha_{n+1}) \in \mathbb{R}^{n+1}$ and $\alpha_{n+1} > 0$, we denote the projection of $\mathbf{v}$ on π by $v = (\alpha_1/\alpha_{n+1}, \dots, \alpha_n/\alpha_{n+1})$. Conversely, if $v \in \mathbb{Q}^n$, then we denote by $\mathbf{v}$ the unique point $\mathbf{v} = (l_1, \dots, l_n, l_{n+1}) \in \mathbb{Z}^{n+1}$ such that $l_1, \dots, l_{n+1}$ are relatively prime, $l_{n+1} > 0$, and $\mathbf{v}$ projects to v. In this case, we say that v is a *rational point* and that the coordinates of $\mathbf{v}$ are the *primitive projective coordinates* of v. Note that this convention differs from the one used in [17], where projective coordinates range from 0 to n, and $\pi = \{x_0 = 1\}$.

An n-dimensional simplex in $\mathbb{R}^n$ is *unimodular* if its vertices $v_1, \dots, v_{n+1}$ are rational and $\mathbf{v}_1, \dots, \mathbf{v}_{n+1}$ constitute a $\mathbb{Z}$-basis of $\mathbb{Z}^{n+1}$. In all this paper, Δ will denote the simplex whose vertices $v_1, \dots, v_{n+1}$ are given, in primitive projective coordinates, by the columns of the following matrix:

$$V = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 1 & \cdots & 1 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}.$$

More precisely, all entries of V are 0, except those in position ij, with either ($i = n+1$) or ($j \geq 2$ and $i + j \leq n + 2$), that have value 1. Clearly Δ is unimodular.

Consider now the following $(n+1)\times(n+1)$ matrices:

$$A_0 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Here, all entries of A_0 and A_1 are 0, except $(A_0)_{11}$, $(A_1)_{21}$, and all elements in position $1(n+1)$, $2(n+1)$, $(j+1)j$, for $2 \le j \le n$, that have value 1. Let Δ_0, Δ_1 be the unimodular simplexes whose vertices are given, in projective coordinates, by the columns of VA_0 and VA_1, respectively. For $a = 0, 1$, the matrix $M_a = VA_a^{-1}V^{-1} \in \mathrm{GL}_{n+1}(\mathbb{Z})$ expresses, in projective coordinates, a fractional-linear homeomorphism —with a slight abuse of notation, again denoted by M_a— from Δ_a to Δ as follows: if $x = (x_1 \cdots x_n)^{tr} \in \Delta_a$, then the projective coordinates of $M_a(x)$ are $M_a(x_1 \cdots x_n\, 1)^{tr}$. Note that $\Delta = \Delta_0 \cup \Delta_1$ and $M_0 = M_1$ on $\Delta_0 \cap \Delta_1$. Indeed, the $(n-1)$-simplex $\Delta_0 \cap \Delta_1$ has vertices given by the columns of VA' (where A' is the $(n+1)\times n$ matrix obtained from either A_0 or A_1 by removing the first column) and $M_0VA' = M_1VA'$. We remark here, for future reference in the proof of Proposition 2.2, that $M_0[\Delta_0 \cap \Delta_1] = M_1[\Delta_0 \cap \Delta_1]$ is the $(n-1)$-dimensional face of Δ whose vertices are $v_2, \dots, v_{n+1}$ (just consider the columns of $VA_0^{-1}A'$). The continuous piecewise-fractional map $M : \Delta \to \Delta$ defined by $M(x) = M_a(x)$, for $x \in \Delta_a$, is the *Mönkemeyer map*. A simple matrix computation shows that $\Delta_0 = \{x \in \Delta : x_1 + x_n \le 1\}$ and that, in affine coordinates,

$$M(x_1, x_2, \dots, x_n) = \begin{cases} \Big(\dfrac{x_1}{1-x_n}, \dfrac{x_1 - x_n}{1-x_n}, \dots, \dfrac{x_{n-1}-x_n}{1-x_n}\Big), & \text{if } x_1 + x_n \le 1; \\[2ex] \Big(\dfrac{1-x_n}{x_1}, \dfrac{x_1 - x_n}{x_1}, \dots, \dfrac{x_{n-1}-x_n}{x_1}\Big), & \text{if } x_1 + x_n \ge 1. \end{cases}$$

For $a = 0, 1$, let now B_a be the $(n+1)\times(n+1)$ matrix which is identical to A_a except for the last column, where the two 1's are replaced by $1/2$. The matrices V and VB_a agree in the last row $(1 \cdots 1\, 1)$. Therefore, the product of the first one with the inverse of the second, i.e., the matrix $T_a = VB_a^{-1}V^{-1}$, has last row $(0 \cdots 0\, 1)$ and determines an affine map $T_a : \Delta_a \to \Delta$ as follows: if $x = (x_1 \cdots x_n)^{tr} \in \Delta_a$ and $y = (y_1 \cdots y_n)^{tr} = T_a(x)$, then $T_a(x_1 \cdots x_n\, 1)^{tr} = (y_1 \cdots y_n\, 1)^{tr}$. As above, $T_0 = T_1$ on $\Delta_0 \cap \Delta_1$. The continuous piecewise-affine map $T : \Delta \to \Delta$ defined by $T(x) = T_a(x)$, for $x \in \Delta_a$, is the *tent map*. In affine coordinates, T is expressed by

$$T(x_1, x_2, \dots, x_n) = \begin{cases} (x_1 + x_n, x_1 - x_n, \dots, x_{n-1} - x_n), & \text{if } x_1 + x_n \le 1; \\ (2 - x_1 - x_n, x_1 - x_n, \dots, x_{n-1} - x_n), & \text{if } x_1 + x_n \ge 1. \end{cases}$$

Note that, for $n = 1$, the Mönkemeyer map and the tent map defined above coincide with the Farey map and the tent map of §1. The following theorem is then an n-dimensional generalization of Proposition 1.1.

Theorem 2.1. *There exists a unique homeomorphism* $\Phi : \Delta \to \Delta$ *such that* $T = \Phi \circ M \circ \Phi^{-1}$.

The rest of this section is devoted to the proof of Theorem 2.1; we first prove the existence of Φ, then its uniqueness. Recall that a *[rational] simplicial complex*

in $\mathbb{R}^n$ is a finite set $\mathcal{C}$ of simplexes in $\mathbb{R}^n$ such that: (1) all vertices of all simplexes in $\mathcal{C}$ are rational; (2) if $\Gamma \in \mathcal{C}$ and Σ is a face of Γ, then $\Sigma \in \mathcal{C}$; (3) every two simplexes intersect in a common face. The *support* of $\mathcal{C}$ is the set-theoretic union $|\mathcal{C}|$ of all simplexes in $\mathcal{C}$. A complex $\mathcal{C}$ *refines* a complex $\mathcal{D}$, written $\mathcal{C} \geq \mathcal{D}$, if $|\mathcal{C}| = |\mathcal{D}|$ and every simplex of $\mathcal{C}$ is contained in some simplex of $\mathcal{D}$. The *mesh* of $\mathcal{C}$, written $\operatorname{mesh}(\mathcal{C})$ is the maximum diameter of the simplexes in $\mathcal{C}$ or, equivalently [12, Corollary 5.18], the maximum length of the 1-simplexes in $\mathcal{C}$.

The set $\mathcal{F}_1$ of all faces of Δ_0 and Δ_1 is a simplicial complex supported in Δ. For short, we list only the maximal (w.r.t. the relation of being a face) simplexes, thus writing $\mathcal{F}_1 = \{\Delta_0, \Delta_1\}$; we also write $\mathcal{F}_0 = \{\Delta\}$. For every finite sequence $a_0, \dots, a_{t-1} \in \{0,1\}$, we define by recursion

$$\begin{aligned}\Delta_{a_0 \dots a_{t-1}} &= \Delta_{a_0} \cap M^{-1} \Delta_{a_1 \dots a_{t-1}} \\ &= \{x : x \in \Delta_{a_0} \;\&\; M(x) \in \Delta_{a_1} \;\&\; M^2(x) \in \Delta_{a_2} \;\&\; \cdots \;\&\; M^{t-1}(x) \in \Delta_{a_{t-1}}\},\end{aligned}$$

and we call $\mathcal{F}_t = \{\Delta_{a_0 \dots a_{t-1}} : a_0, \dots, a_{t-1} \in \{0,1\}\}$ the *time-t partition* for M.

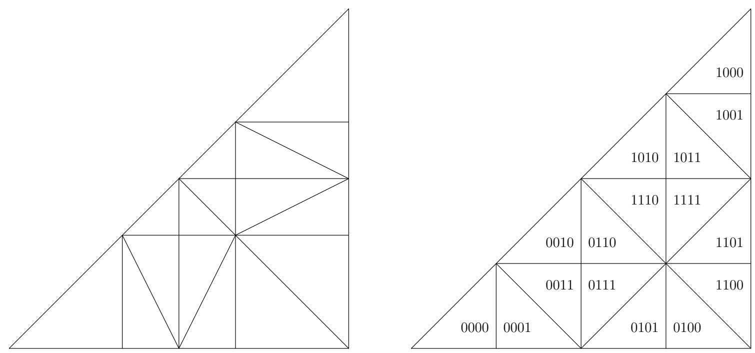
Proposition 2.2. *Every $\mathcal{F}_t$ is a simplicial complex, whose maximal elements are the 2^t n-simplexes $\Delta_{a_0 \dots a_{t-1}}$. For every $t \geq 0$, the complex $\mathcal{F}_{t+1}$ refines $\mathcal{F}_t$.*

Proof. Let $\psi_a = M_a^{-1} : \Delta \to \Delta_a$ be the two inverse branches of M, for $a = 0, 1$. Note that $\Delta_{a_0 \dots a_{t-1}} = \psi_{a_0} \circ \psi_{a_1} \circ \cdots \circ \psi_{a_{t-1}}[\Delta]$. Indeed, this is true for $t = 1$, and follows by induction otherwise, since

$$\begin{aligned}\Delta_{a_0 \dots a_{t-1}} &= \Delta_{a_0} \cap M^{-1} \Delta_{a_1 \dots a_{t-1}} \\ &= \Delta_{a_0} \cap \left[\psi_0[\Delta_{a_1 \dots a_{t-1}}] \cup \psi_1[\Delta_{a_1 \dots a_{t-1}}]\right] \\ &= \psi_{a_0}[\Delta_{a_1 \dots a_{t-1}}].\end{aligned}$$

We now proceed by induction: $\mathcal{F}_0$ and $\mathcal{F}_1$ are simplicial complexes supported on Δ, with $\mathcal{F}_1 \geq \mathcal{F}_0$. Assuming that $\mathcal{F}_t$ is such a complex, then the elements of $\mathcal{F}_{t+1}$ are given by $\psi_0[\mathcal{F}_t] \cup \psi_1[\mathcal{F}_t]$, where $\psi_a[\mathcal{F}_t]$ is the set of all ψ_a-images of the elements of $\mathcal{F}_t$. Since ψ_a is fractional-linear, $\psi_a[\mathcal{F}_t]$ is a simplicial complex supported in Δ_a. It is therefore sufficient to show that $\psi_0[\mathcal{F}_t]$ and $\psi_1[\mathcal{F}_t]$ agree (i.e., induce the same complex) on the intersection $\Delta_0 \cap \Delta_1$. This fact is true because, as we remarked in the course of the definition of the Mönkemeyer map, M_0 and M_1 agree on $\Delta_0 \cap \Delta_1$, and provide a fractional-linear homeomorphism between $\Delta_0 \cap \Delta_1$ and the $(n-1)$-dimensional face Λ of Δ whose vertices are $v_2, \dots, v_{n+1}$. Therefore ψ_0 and ψ_1 agree on Λ. This implies that $\psi_0[\mathcal{F}_t]$ and $\psi_1[\mathcal{F}_t]$ induce the same complex on $\Delta_0 \cap \Delta_1$, namely the ψ_0-image, which is also the ψ_1-image, of the complex induced by $\mathcal{F}_t$ on Λ. The fact that $\mathcal{F}_{t+1}$ refines $\mathcal{F}_t$ is immediate, since every maximal simplex $\Delta_{a_0 \dots a_{t-1} a_t}$ is contained in $\Delta_{a_0 \dots a_{t-1}}$. □

We construct analogously the time-t partition $\mathcal{B}_t$ for T. Namely, we let $\mathcal{B}_0 = \mathcal{F}_0$, $\Gamma_0 = \Delta_0$, $\Gamma_1 = \Delta_1$, and $\Gamma_{a_0 \dots a_{t-1}} = \Gamma_{a_0} \cap T^{-1} \Gamma_{a_1 \dots a_{t-1}}$. The analogue of Proposition 2.2 holds verbatim, and we have simplicial complexes $\mathcal{B}_t = \{\Gamma_{a_0 \dots a_{t-1}} : a_0, \dots, a_{t-1} \in \{0,1\}\}$, with $\mathcal{B}_{t+1}$ refining $\mathcal{B}_t$. An obvious induction on t shows that there exists a unique combinatorial isomorphism from $\mathcal{F}_t$ to $\mathcal{B}_t$ that fixes the vertices of Δ. At the level of maximal simplexes, the isomorphism is given by $\Delta_{a_0 \dots a_{t-1}} \mapsto \Gamma_{a_0 \dots a_{t-1}}$. We draw a picture of $\mathcal{F}_4$ and $\mathcal{B}_4$, for $n = 2$, labeling the 2-simplex $\Gamma_{a_0 a_1 a_2 a_3} \in \mathcal{B}_4$ by $a_0 a_1 a_2 a_3$.



Let $\{0,1\}^{\mathbb{N}}$ be the *Cantor space*, i.e., the set of all infinite sequences $\bar{a} = a_0a_1a_2\ldots$ of elements of $\{0,1\}$, endowed with the product topology. For $t \geq 1$, we write $\bar{a} \restriction t$ for $a_0a_1\ldots a_{t-1}$, and we let $[a_0\ldots a_{t-1}]$ be the *cylinder* $\{\bar{b} : \bar{a} \restriction t = \bar{b} \restriction t\}$; we extend this convention by setting $\Delta_{\bar{a}\restriction 0} = \Gamma_{\bar{a}\restriction 0} = \Delta$ and $[a_0\ldots a_{-1}] = \{0,1\}^{\mathbb{N}}$.

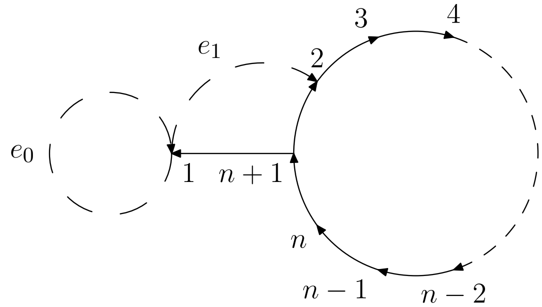
Lemma 2.3. *For every $\bar{a} \in \{0,1\}^{\mathbb{N}}$, the intersection $\bigcap_{t\geq 0}\Delta_{\bar{a}\restriction t}$ is a singleton, and the intersection $\bigcap_{t\geq 0}\Gamma_{\bar{a}\restriction t}$ is a singleton as well.*

Proof. The first statement amounts to saying that the Mönkemeyer algorithm is topologically convergent [17, Definition 9]: this fact is proved in [14, Satz 10] as well as in [17, Lemma 19]. Note that [14, Satz 10] assumes that $\bigcap_{t\geq 0}\Delta_{\bar{a}\restriction t}$ contains a point whose coordinates are not all rational. But this is not a restriction since, if $\bigcap_{t\geq 0}\Delta_{\bar{a}\restriction t}$ contained two distinct points p,q then, by convexity, it would contain all the points in the line segment connecting p with q, and hence a point whose coordinates are not all rational.

In order to prove the second statement note that the vertices of $\Gamma_{\bar{a}\restriction t}$ are given, in projective coordinates, by the columns of $VB_{a_0}\cdots B_{a_{t-1}}$. Let $\mathfrak{K}$ be the set of all $(n+1)\times(n+1)$ column-stochastic matrices (i.e., all nonnegative matrices having the property that the entries in each column sum up to 1). Observe that $\mathfrak{K}$ is a compact submonoid of $(\mathrm{Mat}_{(n+1)\times(n+1)}\,\mathbb{R},\cdot,\mathrm{Id})$. Let $B_{\bar{a}\restriction t} = B_{a_0}\cdots B_{a_{t-1}} \in \mathfrak{K}$. We will apply [6, Theorem 6.1] to show that, for every $\bar{a} \in \{0,1\}^{\mathbb{N}}$, the limit $\tilde{B} = \lim_{t\to\infty} B_{\bar{a}\restriction t}$ exists (necessarily in $\mathfrak{K}$, since the latter is closed), and all columns of $\tilde{B}$ are equal. Recall that a column-stochastic matrix $C = C_{ij}$ is (j_1,j_2)-*scrambling* if there exists a row index i such that C_{ij_1} and C_{ij_2} are both > 0; C is *scrambling* if it is (j_1,j_2)-scrambling for every pair (j_1,j_2) of columns indices [10]. By [6, Theorem 6.1], it will be sufficient to prove the following:

(A) there exists $s > 0$ such that all products of s matrices from $\{B_0,B_1\}$ (repetitions allowed) are scrambling.

It is simpler to argue on the incidence graphs $G(B_0)$ and $G(B_1)$ associated to B_0 and B_1. The graph $G(B_0)$ has $n+1$ vertices and there is a directed edge connecting the jth vertex to the ith iff $(B_0)_{ij} > 0$; similarly for $G(B_1)$. We combine $G(B_0)$ and $G(B_1)$ in a single graph G as in the following picture, with the understanding that in $G(B_0)$ the edge e_0 is activated and the edge e_1 is discarded, and conversely in $G(B_1)$.



We will deduce property (A) from the existence of a winning strategy for a certain game on G. The game starts with two Lovers sitting in distinct vertices. A move of the game consists of the following: first, the Enemy chooses which of the two edges e_0 and e_1 is to be active at that move, and then each Lover moves one step along an edge departing from his/her current vertex. The Lovers win the game if after finitely many moves they end up in the same vertex.

Claim. Regardless of the initial position, the Lovers win in at most $(n+1)n/2$ moves.

Assuming the Claim, let us prove (A). We take $s = (n+1)n/2$, and we fix a product $B = B_{a_0} \cdots B_{a_{s-1}}$ of s matrices from $\{B_0, B_1\}$. No column in B_0 or in B_1 is identically 0; therefore, for every (j_1, j_2)-scrambling matrix C, both B_0C and B_1C are (j_1, j_2)-scrambling. Choose column indices j_1, j_2; by the above, we can assume $j_1 \neq j_2$. Consider the game in which the Lovers apply the winning strategy and start in position j_1 and j_2, while the Enemy activates the edge $e_{a_{s-r}}$ at the rth move ($r \geq 1$). By the Claim, this game ends after $1 \leq t \leq s$ moves, leaving the Lovers in the same vertex i. This implies that there exists a path in G connecting j_1 to i, and such that the rth edge in the path is an edge of $G(B_{a_{s-r}})$. By the elementary properties of the incidence graphs of nonnegative square matrices, the ij_1th entry of $B_{a_{s-t}} \cdots B_{a_{s-2}} B_{a_{s-1}}$ is > 0. Analogously, the ij_2th entry is > 0; hence $B_{a_{s-t}} \cdots B_{a_{s-2}} B_{a_{s-1}}$ is (j_1, j_2)-scrambling, and so is B. Since j_1 and j_2 are arbitrary, B is scrambling.

Proof of Claim. Given any vertex p of G, there exists a unique vertex path $p = p_0, p_1, p_2, \ldots$ such that, for every $i \geq 1$, there is an oriented edge in G, different from both e_0 and e_1, which connects p_i to p_{i-1}. Let us call such a path a *backward path*. The length of a finite backward path $p_0, p_1, p_2, \ldots, p_r$ is r, its origin is p_0, and its endpoint is p_r. Let $\mathcal{V}$ be the set, of cardinality $(n+1)n/2$, whose elements are all unordered pairs of distinct vertices of G. If $\{p, q\} \in \mathcal{V}$, then the *gap* $g(p, q)$ of $\{p, q\}$ is the minimal length of a backward path whose origin is one of the vertices p, q, and whose endpoint is the other vertex. The origin of such a path is the *leading vertex* of $\{p, q\}$, and the *defect* $d(p, q)$ of $\{p, q\}$ is 0 if the leading vertex is 1, and is $g(1, \text{leading vertex})$ otherwise. The leading vertex, and the numbers $1 \leq g(p, q) \leq n$ and $0 \leq d(p, q) \leq n$, are uniquely determined by the pair, with the exception of the case in which n is even, p and q are both $\neq 1$, and $g(p, q) = n/2$. In this case, we define the leading vertex to be the vertex whose gap from 1 is minimal, and we define the defect accordingly. Let $\mathcal{T}$ be the set $\{1, \ldots, n\} \times \{0, \ldots, n\}$, ordered lexicographically: $(g, d) \prec (g', d')$ iff $g < g'$ or ($g = g'$ and $d < d'$). The

map $\chi : \mathcal{V} \to \mathcal{T}$ defined by $\chi\{p,q\} = \big(g(p,q), d(p,q)\big)$ is injective: indeed $d(p,q)$ determines uniquely the leading vertex of the pair (start from 1 and go backwards $d(p,q)$ steps in G, never using the edges e_0 and e_1), and then $g(p,q)$ determines the other vertex.

Assume now that at a certain stage of the game the Lovers are in distinct vertices p, q, and consider the following strategy.

(a) If $\{p,q\} = \{1, n+1\}$, then the Lovers win at the next move, either by meeting at 1 (if the Enemy chooses to activate e_0) or by meeting at 2 (if the Enemy activates e_1).

(b) Otherwise, if $n+1 \notin \{p,q\}$, then each Lover follows the unique edge at his disposal.

(b) Otherwise, without loss of generality $p = n+1$ and $q \notin \{1, n+1\}$. Then the Lover at q follows the unique edge at his disposal, while the Lover at p moves to 1 provided that p is the leading vertex of the pair $\{p,q\}$, otherwise moves to 2.

Let p', q' be the vertices occupied by the Lovers at the next step, and assume that the game is not finished yet (hence case (a) did not apply, and $p' \neq q'$). An easy case analysis, distinguishing the three cases (i) the leading vertex is 1, (ii) the leading vertex is $n+1$, and (iii) the leading vertex is in $\{2, \ldots, n\}$, shows that $\chi(p', q') \prec \chi(p,q)$. In other words, at each step either the gap of the pair decreases, or the gap stays the same and the defect decreases. Since $\chi[\mathcal{V}]$ has finite size and is totally ordered by $\prec$, the length of the longest possible game, final winning move included, coincides with this size, namely $(n+1)n/2$. □

Note that the bound in the proof of Lemma 2.3 is sharp: for $n = 2$ the matrix A_0A_1 is not scrambling, and for $n = 3$ the matrix $A_0A_0A_0A_1A_0$ is not scrambling either.

Corollary 2.4. *Both* $\lim_{t\to\infty} \operatorname{mesh}(\mathcal{F}_t)$ *and* $\lim_{t\to\infty} \operatorname{mesh}(\mathcal{B}_t)$ *exist and have value* 0.

Proof. Suppose that statement is false for, say, the Farey complexes. Then there exists $\varepsilon > 0$ such that, for every t, the set

$$\mathcal{S}_t = \{a_0a_1 \ldots a_{t-1} : \text{the diameter of } \Delta_{a_0 \ldots a_{t-1}} \text{ is } \geq \varepsilon\}$$

is not empty. If $a_0a_1 \ldots a_{t-1} \in \mathcal{S}_t$ and $0 < k \leq t$, then $a_0a_1 \ldots a_{k-1} \in \mathcal{S}_k$, so the union of all $\mathcal{S}_t$'s is an infinite subtree of the full binary tree. By König's Lemma [19, Lemma 3.3.19] this subtree has an infinite branch $\bar{a} \in \{0,1\}^{\mathbb{N}}$. This contradicts Lemma 2.3. □

As a side remark note that, for $n = 2$, all the 2^t triangles in $\mathcal{B}_t$ are congruent, because the 2×2 matrices obtained from T_0^{-1} and T_1^{-1} by removing the third row and the third column are both of the form $1/\sqrt{2} \cdot$ (an orthogonal matrix). This is no longer true for $n \geq 3$.

Given $\bar{a} \in \{0,1\}^{\mathbb{N}}$, let $\varphi(\bar{a})$ be the unique point in $\bigcap_{t \geq 0} \Delta_{\bar{a}\restriction t}$, and let $\upsilon(\bar{a})$ be the unique point in $\bigcap_{t \geq 0} \Gamma_{\bar{a}\restriction t}$.

Lemma 2.5. *The mappings* $\varphi, \upsilon : \{0,1\}^{\mathbb{N}} \to \Delta$ *are continuous, surjective, and have the same fibers (i.e.,* $\varphi(\bar{a}) = \varphi(\bar{b})$ *iff* $\upsilon(\bar{a}) = \upsilon(\bar{b})$*).*

Proof. Clearly φ is surjective and we have

$$\Delta_{a_0\dots a_{t-1}} \subseteq \varphi\bigl[[a_0 \dots a_{t-1}]\bigr]; \tag{$*$}$$
$$[a_0 \dots a_{t-1}] \subseteq \varphi^{-1}\Delta_{a_0\dots a_{t-1}}. \tag{$**$}$$

If $U \subseteq \Delta$ is open, then we have

$$\varphi^{-1}U = \bigcup\{[a_0 \dots a_{t-1}] : \Delta_{a_0\dots a_{t-1}} \subseteq U\}. \tag{$***$}$$

Indeed, the $\supseteq$ inclusion is immediate from $(**)$. On the other hand, let $\varphi(\bar{a}) \in U$. Then $(\Delta \setminus U) \cap \bigcap_{t\geq 0} \Delta_{\bar{a}\restriction t} = \emptyset$, and hence by compactness there exists $t \geq 0$ such that $\Delta_{\bar{a}\restriction t} \subseteq U$. Therefore $\bar{a}$ belongs to the right-hand side of $(***)$, and equality follows. Since the right-hand side of $(***)$ is open in the Cantor space, φ is continuous. Exactly the same proof shows that υ is surjective and continuous as well.

We assume now that $\bar{a}, \bar{b}$ are such that $\varphi(\bar{a}) = \varphi(\bar{b})$, and prove $\upsilon(\bar{a}) = \upsilon(\bar{b})$. By hypothesis, for every $t \geq 0$ we have $\Delta_{\bar{a}\restriction t} \cap \Delta_{\bar{b}\restriction t} \neq \emptyset$, and hence $\Delta_{\bar{a}\restriction t}$ and $\Delta_{\bar{b}\restriction t}$ intersect in a common nonempty face. As observed above, $\mathcal{F}_t$ and $\mathcal{B}_t$ are combinatorially isomorphic; therefore, for every t, $\Gamma_{\bar{a}\restriction t}$ and $\Gamma_{\bar{b}\restriction t}$ intersect in a common nonempty face as well. Again by compactness, $\bigcap_{t\geq 0}(\Gamma_{\bar{a}\restriction t} \cap \Gamma_{\bar{b}\restriction t}) \neq \emptyset$. Since by definition $\bigcap_{t\geq 0} \Gamma_{\bar{a}\restriction t} = \{\upsilon(\bar{a})\}$ and $\bigcap_{t\geq 0} \Gamma_{\bar{b}\restriction t} = \{\upsilon(\bar{b})\}$, we have $\upsilon(\bar{a}) = \upsilon(\bar{b})$. Clearly the rôle of φ and υ can be reversed, and it follows that φ and υ have the same fibers. $\square$

Let $\equiv$ be the equivalence relation on the Cantor space defined by $\bar{a} \equiv \bar{b}$ iff $\varphi(\bar{a}) = \varphi(\bar{b})$ iff $\upsilon(\bar{a}) = \upsilon(\bar{b})$. Let $\chi : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}/\equiv$ be the quotient mapping, and endow $\{0,1\}^{\mathbb{N}}/\equiv$ with the quotient topology: V is open in $\{0,1\}^{\mathbb{N}}/\equiv$ iff $\chi^{-1}V$ is open in $\{0,1\}^{\mathbb{N}}$. We have an obvious factorization in continuous mappings

$$\begin{tikzcd}
 & \{0,1\}^{\mathbb{N}} \arrow[dl, "\varphi"'] \arrow[d, "\chi"] \arrow[dr, "\upsilon"] & \\
\Delta & \{0,1\}^{\mathbb{N}}/\equiv \arrow[l, "\bar{\varphi}"] \arrow[r, "\bar{\upsilon}"'] & \Delta
\end{tikzcd}$$

The quotient space $\{0,1\}^{\mathbb{N}}/\equiv$ is compact, and Δ is Hausdorff. Hence the continuous bijections $\bar{\varphi}$ and $\bar{\upsilon}$ are both homeomorphisms.

Definition 2.6. We define $\Phi : \Delta \to \Delta$ as the homeomorphism $\Phi = \bar{\upsilon} \circ \bar{\varphi}^{-1}$. Equivalently, $\Phi(p) = \upsilon(\bar{a})$, for any $\bar{a}$ such that $\varphi(\bar{a}) = p$.

For every $a_0, \dots, a_{t-1}$, the homeomorphism Φ restricts to a bijection from $\Delta_{a_0\dots a_{t-1}}$ to $\Gamma_{a_0\dots a_{t-1}}$; this follows from $(*)$ and $(**)$ in the proof of Lemma 2.5 and the corresponding inclusions for υ. For every $t > 1$ we have $M[\Delta_{a_0\dots a_{t-1}}] = M \circ \psi_{a_0}[\Delta_{a_1\dots a_{t-1}}] = \Delta_{a_1\dots a_{t-1}}$; this makes sense also for $t = 1$, since $M[\Delta_{a_0}] = \Delta$. Analogously we have $T[\Gamma_{a_0\dots a_{t-1}}] = \Gamma_{a_1\dots a_{t-1}}$. Denoting by S the shift map on

$\{0,1\}^{\mathbb{N}}$ (i.e., $S(a_0a_1a_2\dots) = (a_1a_2a_3\dots)$), it easily follows that the following diagram commutes:

$$\begin{tikzcd}
\Delta \arrow[r, "M"] & \Delta \\
\{0,1\}^{\mathbb{N}} \arrow[u, "\varphi"] \arrow[r, "S"] \arrow[d, "\upsilon"'] & \{0,1\}^{\mathbb{N}} \arrow[u, "\varphi"'] \arrow[d, "\upsilon"] \\
\Delta \arrow[r, "T"'] & \Delta
\arrow[from=1-1, to=3-1, bend right=60, "\Phi"']
\arrow[from=1-2, to=3-2, bend left=60, "\Phi"]
\end{tikzcd}$$

By chasing the diagram, one sees immediately that $T = \Phi \circ M \circ \Phi^{-1}$, as required.

We now show the uniqueness of Φ, by assuming that Ψ is a homeomorphism of Δ such that $T = \Psi \circ M \circ \Psi^{-1}$ and proving that $\Psi = \Phi$. Observe that the boundary $\partial\Delta$ of Δ is characterized —in purely topological terms, with no reference to the immersion of Δ in $\mathbb{R}^n$— as the set of points $p \in \Delta$ whose removal leaves $\Delta \setminus \{p\}$ contractible. Therefore, $\Phi[\partial\Delta] = \Psi[\partial\Delta] = \partial\Delta$. Let Σ be the set-theoretic union of the proper faces of Δ_0 and Δ_1; we have $\Sigma = M^{-1}\partial\Delta = T^{-1}\partial\Delta$ and, as a consequence, $\Sigma = \Phi^{-1}\Sigma = \Psi^{-1}\Sigma$. Indeed, e.g., $p \in \Sigma$ iff $M(p) \in \partial\Delta$ iff $\Psi \circ M(p) \in \partial\Delta$ iff $T \circ \Psi(p) \in \partial\Delta$ iff $\Psi(p) \in T^{-1}\partial\Delta = \Sigma$ iff $p \in \Psi^{-1}\Sigma$.

Note that $\Phi(v_1) = \Psi(v_1) = v_1$. Indeed, v_1 is a point in $\partial\Delta$ which is fixed by M, and therefore both $\Phi(v_1)$ and $\Psi(v_1)$ must be points in $\partial\Delta$ which are fixed by T. But there is only one such point, namely v_1 itself. Observe now that Δ_0 can be characterized as the set of points in Δ that can be connected to v_1 by a continuous path whose relative interior does not intersect Σ. Since Φ and Ψ are homeomorphisms, both fixing Σ globally, we can safely conclude that $\Phi[\Delta_a] = \Psi[\Delta_a]$, for $a = 0, 1$.

Let now p be any point of Δ, and choose $\bar{a}$ such that $\varphi(\bar{a}) = p$ and $\upsilon(\bar{a}) = \Phi(p)$. For every $t \geq 0$ we have $M^t(p) \in \Delta_{a_t} = \Gamma_{a_t}$, and therefore $\Psi^{-1} \circ T^t\big(\Psi(p)\big) \in \Gamma_{a_t}$, i.e., $T^t\big(\Psi(p)\big) \in \Psi[\Gamma_{a_t}] = \Gamma_{a_t}$. Hence, by definition of υ, $\Psi(p) = \upsilon(\bar{a}) = \Phi(p)$. This concludes the proof of Theorem 2.1.

3. Fractal structure, periodicity and singularity

In this section we will discuss how properties §1(1)–(5) of the classical Minkowski function generalize to our n-dimensional setting. Basically, all properties continue to hold, with the exception of §1(2), whose full validity turns out to be an open problem. Let us first treat §1(4).

Proposition 3.1. *Let $t \geq 0$, let $\Xi \in \mathcal{F}_t$, and let Λ be the simplex in $\mathcal{B}_t$ corresponding to Ξ under the combinatorial isomorphism defined before Lemma 2.3. Then Φ restricts to a homeomorphism between Ξ and Λ. Moreover, for every $\Delta_{a_0\dots a_{t-1}} \in \mathcal{F}_t$ we have*

$$\Phi = (T_{a_{t-1}} \circ \cdots \circ T_{a_0}) \circ (\Phi \restriction \Delta_{a_0\dots a_{t-1}}) \circ (M_{a_{t-1}} \circ \cdots \circ M_{a_0})^{-1}.$$

Proof. We can give an equivalent definition of Φ and Φ^{-1} as follows. For each t, we define a simplicial homeomorphism $\Phi_t : \Delta \to \Delta$ by first mapping the vertices of $\mathcal{F}_t$ to the vertices of $\mathcal{B}_t$ according to the combinatorial isomorphism, and then using barycentric coordinates to extend Φ_t to all of Δ. More precisely, if $\Delta_{a_0\dots a_{t-1}}$ has vertices $w_1, \dots, w_{n+1}$ and $\Delta_{a_0\dots a_{t-1}} \ni p = \sum \alpha_i w_i$ in barycentric coordinates, then

$\Phi_t(p) = \sum \alpha_i \Phi_t(w_i)$. Using Corollary 2.4, one sees easily that $\Phi = \lim_{t\to\infty} \Phi_t$ and $\Phi^{-1} = \lim_{t\to\infty} \Phi_t^{-1}$ in the topology of uniform convergence. Now, for every $p \in \Xi$ and every $k \geq t$, we have $\Phi_k(p) \in \Lambda$; since Λ is closed, $\Phi(p) = \lim_{k\to\infty} \Phi_k(p) \in \Lambda$. Hence $\Phi[\Xi] \subseteq \Lambda$, and the same argument applied to Φ^{-1} shows that $\Phi[\Xi] = \Lambda$.

The mappings

$$M_{a_{t-1}} \circ \cdots \circ M_{a_0} = M \upharpoonright \Delta_{a_0 \ldots a_{t-1}} : \Delta_{a_0 \ldots a_{t-1}} \to \Delta,$$

and

$$T_{a_{t-1}} \circ \cdots \circ T_{a_0} = T \upharpoonright \Gamma_{a_0 \ldots a_{t-1}} : \Gamma_{a_0 \ldots a_{t-1}} \to \Delta,$$

are both homeomorphisms, the former fractional-linear and the latter affine. Our second statement is then immediate, since it amounts to the restriction of the identity $\Phi \circ M = T \circ \Phi$ to $\Delta_{a_0 \ldots a_{t-1}}$. $\square$

Next, §1(1) generalizes to the following proposition.

Proposition 3.2. Φ *is an orientation-preserving homeomorphism.*

Proof. Choose t such that $\mathcal{F}_t$ contains a vertex p in the topological interior Δ° of Δ. Let $\Delta_{a_0 \ldots a_{t-1}} \in \mathcal{F}_t$, and let D be the diagonal matrix whose entries along the main diagonal are those in the last row of $VA_{a_0} \cdots A_{a_{t-1}}$. Then the affine homeomorphism $\Phi_t \upharpoonright \Delta_{a_0 \ldots a_{t-1}}$ defined in the proof of Proposition 3.1 is expressed by the matrix $(VB_{a_0} \cdots B_{a_{t-1}})(VA_{a_0} \cdots A_{a_{t-1}} D^{-1})^{-1}$, which has last row $(0 \cdots 0\, 1)$ and determinant > 0 (because D has positive determinant, and A_a, B_a have determinant of the same sign, for $a \in \{0, 1\}$). This holds for every maximal simplex $\Delta_{a_0 \ldots a_{t-1}}$ in $\mathcal{F}_t$, and it follows that Φ_t is orientation-preserving.

Let now q be the vertex in $\mathcal{B}_t$ corresponding to p. Again q is in Δ°, and $\Phi(p) = \Phi_t(p) = q$. Let $X = \Delta \setminus \{p\}$ and $Y = \Delta \setminus \{q\}$. Then $\Phi \upharpoonright X$ and $\Phi_t \upharpoonright X$ are homeomorphisms from X to Y, and we claim that they are homotopic. Indeed, a homotopy $F : X \times [0,1] \to Y$ is given by $(x, r) = (1-r)\Phi(x) + r\Phi_t(x)$. This works because, assuming $x \in \Delta_{a_0 \ldots a_{t-1}}$, we have by Proposition 3.1 that $\Phi(x)$ and $\Phi_t(x)$ are both in $\Gamma_{a_0 \ldots a_{t-1}} \setminus \{q\}$. Since $\Gamma_{a_0 \ldots a_{t-1}} \setminus \{q\}$ is convex, the image of F is Y. One checks easily that F is continuous, and this establishes our claim.

Note that, given any points $p', q' \in \Delta^\circ$, the homology groups $H_{n-1}(\Delta \setminus \{p'\})$ and $H_{n-1}(\Delta \setminus \{q'\})$ (coefficients in $\mathbb{Z}$) are canonically identifiable, since they are both canonically isomorphic to the relative homology group $H_n(\Delta, \Delta \setminus B)$, where B is any ball in Δ° containing p' and q'. By [11, p. 233], we have that Φ (respectively, Φ_t) is orientation-preserving iff $\Phi \upharpoonright X$ (respectively, $\Phi_t \upharpoonright X$) induces in homology the identity mapping between $H_{n-1}(X)$ and $H_{n-1}(Y)$ (these two infinite cyclic groups canonically identified as above). Since $\Phi \upharpoonright X$ and $\Phi_t \upharpoonright X$ are homotopic, they induce the same isomorphism in homology, and we conclude that Φ is orientation-preserving iff so is Φ_t. $\square$

As remarked at the beginning of this section, a proper generalization of §1(2) is critical. Indeed, the periodicity properties of the various multidimensional continued fraction algorithms are a long-standing open problem. Even for the most studied algorithm, the Jacobi-Perron one [16], [17], it is still unknown whether points $p = (\alpha_1, \ldots, \alpha_n)$ such that $[\mathbb{Q}(\alpha_1, \ldots, \alpha_n) : \mathbb{Q}] \leq n + 1$ are always preperiodic under the piecewise-fractional map associated to the algorithm. The situation for the Mönkemeyer algorithm is no better; we list a few simple facts in order to describe the problem.

Let $p \in \Delta$; the *grand orbit* of p under M is

$$\mathrm{GO}_M(p) = \{q \in \Delta : M^t(p) = M^s(q) \text{ for some } t, s \geq 0\},$$

and its *eventual periodic orbit* is

$$\mathrm{EPO}_M(p) = \{q \in \Delta : q = M^t(p) = M^s(p) \text{ for some } 0 \leq t < s\}.$$

$\mathrm{EPO}_M(p)$ is always a finite set, possibly empty; if it is nonempty, then p is *preperiodic* under M. One defines $\mathrm{GO}_T(p)$ and $\mathrm{EPO}_T(p)$ similarly; of course $\Phi\bigl[\mathrm{GO}_M(p)\bigr] = \mathrm{GO}_T\bigl(\Phi(p)\bigr)$ and $\Phi\bigl[\mathrm{EPO}_M(p)\bigr] = \mathrm{EPO}_T\bigl(\Phi(p)\bigr)$. Let $\mathbb{Z}[1/2] = \{a/2^m \in \mathbb{Q} : a, m \in \mathbb{Z} \text{ and } m \geq 0\}$ be the ring of dyadic rationals. It is a p.i.d., since it is a localization of the p.i.d. $\mathbb{Z}$. For $p = (\alpha_1, \ldots, \alpha_n) \in \Delta$, we write $\mathbb{Q}(p)$ for the field $\mathbb{Q}(\alpha_1, \ldots, \alpha_n)$, and we write $\mathbb{Z}[1/2](p)$ for the $\mathbb{Z}[1/2]$-module $\mathbb{Z}[1/2]\alpha_1 + \cdots + \mathbb{Z}[1/2]\alpha_n + \mathbb{Z}[1/2]$, which is free of rank $\leq n+1$. Since the matrices M_0, M_1 determining M are in $\mathrm{GL}_{n+1}\mathbb{Z}$, we have clearly $\mathbb{Q}(p) = \mathbb{Q}(q)$, for any $q \in \mathrm{GO}_M(p)$. Analogously, the matrices $T_0, T_1 \in \mathrm{Mat}_{n+1}\mathbb{Z}$ determining T are in $\mathrm{GL}_{n+1}\mathbb{Z}[1/2]$, and hence $\mathbb{Z}[1/2](p) = \mathbb{Z}[1/2](q)$, for any $q \in \mathrm{GO}_T(p)$. We call a point $p = (\alpha_1, \ldots, \alpha_n) \in \Delta$ a *rational point* (respectively, a *dyadic point*) if $\alpha_1, \ldots, \alpha_n \in \mathbb{Q}$ (respectively, $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}[1/2]$). In order to prove that Φ determines a 1-1 correspondence between the rational points and the dyadic ones, we need two technical lemmas.

Remember that a nonsingular matrix $H = H_{ij} \in \mathrm{Mat}_{n\times n}\mathbb{Z}$ is in row Hermite Normal Form (HNF) if it is upper triangular, $H_{jj} > 0$ for every j, and $0 \leq H_{ij} < H_{jj}$ for every $1 \leq i < j$. Every nonsingular $A \in \mathrm{Mat}_{n\times n}\mathbb{Z}$ has a unique HNF (i.e., there exists a unique H in HNF and a —unique— $X \in \mathrm{GL}_n\mathbb{Z}$ such that $H = XA$) [5, §2.4.2]. In particular, two nonsingular matrices $A, B \in \mathrm{Mat}_{n\times n}\mathbb{Z}$ have the same HNF iff there exists $X \in \mathrm{GL}_n\mathbb{Z}$ such that $B = XA$; in this case we write $A \sim B$.

Lemma 3.3. *Let $t \geq 1$, and let $a_0, \ldots, a_{t-1} \in \{0, 1\}$. The matrices $T_{a_{t-1}} \cdots T_{a_0}$ and T_0^t have the same HNF.*

Proof. The last row of T_0 and T_1, and hence of all products $T_{a_{t-1}} \cdots T_{a_0}$, is $(0 \cdots 0\, 1)$. Hence we can safely replace T_a with the $n \times n$ matrix Q_a obtained from T_a by removing the last row and the last column. It now suffices to show that $Q_{a_{t-1}} \cdots Q_{a_0}$ and Q_0^t have the same HNF. Direct computation shows that the entries of Q_0 are as follows:

$$(Q_0)_{ij} = \begin{cases} 1, & \text{if } ij = 11, \text{ or } ij = 1n, \text{ or } i = j+1; \\ -1, & \text{if } i \geq 2 \text{ and } j = n; \\ 0, & \text{otherwise.} \end{cases}$$

We have $Q_0^n = 2E_0$, where E_0 is the $n \times n$ identity matrix; in particular, all powers of Q_0^n commute with everything. For $1 \leq r \leq n-1$, denote the HNF of Q_0^r by E_r; we have explicitly:

$$(E_r)_{ij} = \begin{cases} 2, & \text{if } i = j \geq n - r + 1; \\ 1, & \text{if } i = j < n - r + 1, \text{ or } i < j = n - r + 1; \\ 0, & \text{otherwise.} \end{cases}$$

We work by induction on t. Denote by D the $n \times n$ diagonal matrix whose entries along the diagonal are $-1, 1, \ldots, 1$. Since $Q_1 = DQ_0$, we always have $Q_{a_0} \sim Q_0$, and the case $t = 1$ is settled. By inductive hypothesis, assume $Q_{a_{t-1}} \cdots Q_{a_1} \sim Q_0^{t-1}$.

Then $Q_{a_{t-1}} \cdots Q_{a_1} Q_{a_0} \sim Q_0^{t-1} Q_{a_0}$, and we claim that $Q_0^{t-1} Q_{a_0} \sim Q_0^t$. This is immediate if $a_0 = 0$, so we assume $a_0 = 1$. Let $t - 1 = mn + r$, for some $m \geq 0$ and $0 \leq r < n$. Note that $E_r D \sim E_r$; indeed, $E_r D$ is obtainable from E_r by row operations, namely by first forming DE_r and then, if $0 < r$, summing to the first row of DE_r the $(n - r + 1)$th row. Therefore we have

$$Q_0^{t-1} Q_1 = Q_0^{mn} Q_0^r D Q_0 = Q_0^r D Q_0^{mn+1} \sim E_r D Q_0^{mn+1} \sim \\ \sim E_r Q_0^{mn+1} \sim Q_0^r Q_0^{mn+1} = Q_0^t,$$

as claimed. □

Lemma 3.4. *Let $s > 0$, let $a_0, \ldots, a_{s-1} \in \{0, 1\}$, and let $M_* = M_{a_{s-1}} \cdots M_{a_0}$, $T_* = T_{a_{s-1}} \cdots T_{a_0}$. Then:*

(i) *there exists a unique point $q = (\alpha_1, \ldots, \alpha_n) \in \Delta$ such that $(\alpha_1 \cdots \alpha_n\, 1)^{tr}$ is a right eigenvector for M_* whose corresponding eigenvalue ξ is positive; we then have $\mathbb{Q}(\xi) = \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$;*

(ii) *an analogous statement holds for T_*; in this case $\xi = 1$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{Q}$.*

Proof. Recall that we are identifying $\mathbb{R}^n$ with the plane $\{x_{n+1} = 1\}$ in $\mathbb{R}^{n+1}$, the latter viewed as a space of column vectors. Accordingly, given a simplex Σ in $\mathbb{R}^n$, we write $\mathbb{R}_{\geq 0}\Sigma$ for the polyhedral cone $\{r(\alpha_1 \cdots \alpha_n\, 1)^{tr} : r \geq 0 \text{ and } (\alpha_1, \ldots, \alpha_n) \in \Sigma\}$. Let $\bar{a} \in \{0, 1\}^{\mathbb{N}}$ be defined by $a_t = a_{t \pmod s}$. Then, for every $k \geq 0$, we have $M_*^{-k}[\mathbb{R}_{\geq 0}\Delta] = \mathbb{R}_{\geq 0}\Delta_{\bar{a}\restriction ks}$. By Lemma 2.3, the intersection $\bigcap_{k \geq 0} \Delta_{\bar{a}\restriction ks}$ is the singleton of a point $q = (\alpha_1, \ldots, \alpha_n)$. This immediately implies that M_*^{-1} has (up to scalar multiples) a unique eigenvector in $\mathbb{R}_{\geq 0}\Delta$, namely $(\alpha_1 \cdots \alpha_n\, 1)^{tr}$, whose corresponding eigenvalue ξ^{-1} is positive, and the first statement in (i) follows. Observe now that $V^{-1} M_*^{-1} V = A_{a_0} \cdots A_{a_{s-1}}$ is a nonnegative matrix. By the Perron-Frobenius theory (see, e.g., [8, Chapter III]), there exists a permutation matrix P such that $P^{-1} A_{a_0} \cdots A_{a_{s-1}} P$ has the block form

$$\begin{pmatrix} E_1 & 0 & 0 & 0 \\ * & E_2 & 0 & 0 \\ * & * & \ddots & 0 \\ * & * & * & E_r \end{pmatrix}$$

with each E_i a nonsingular primitive matrix. The $m \times m$ matrix E_r has a dominant simple eigenvalue $\rho > 0$ whose corresponding one-dimensional right eigenspace is spanned by a strictly positive column vector $(\beta_1 \cdots \beta_m)^{tr} \in \mathbb{Q}(\rho)^m$. Since M_*^{-1} and $A_{a_0} \cdots A_{a_{s-1}}$ are conjugate by V, and the V-image of the positive orthant $\mathbb{R}^{n+1}_{\geq 0}$ of $\mathbb{R}^{n+1}$ is $\mathbb{R}_{\geq 0}\Delta$, we have from the first part of the proof that $A_{a_0} \cdots A_{a_{s-1}}$ has exactly (up to scalar multiples) one eigenvector in the positive orthant whose corresponding eigenvalue is positive. This eigenvector is necessarily $P(0 \cdots 0\, \beta_1 \cdots \beta_m)^{tr}$, and $\rho = \xi^{-1}$. Going back to $\mathbb{R}_{\geq 0}\Delta$, we have that $(\alpha_1 \cdots \alpha_n\, 1)^{tr}$ is a real multiple of $VP(0 \cdots 0\, \beta_1 \cdots \beta_m)^{tr}$. Hence $(\alpha_1 \cdots \alpha_n\, 1)^{tr}$ is a real multiple of a vector in $\mathbb{Q}(\rho)^{n+1}$, and therefore $\alpha_1, \ldots, \alpha_n \in \mathbb{Q}(\rho) = \mathbb{Q}(\xi)$; since M_* has integer entries, $\xi \in \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$. The same proof shows (ii); in this case $\xi = 1$ because the last row of T_* is $(0 \cdots 0\, 1)$. □

Lemma 3.4(i) should be compared with [4, Theorem 3.1] and [17, Theorem 42]. In both cases it is proved that a purely periodic point has coordinates in a field of the form $\mathbb{Q}(\xi)$, for ξ an eigenvalue of an appropriate periodicity matrix. However,

in the first case it is required that ξ has maximal degree $n+1$, while in the second the periodicity matrix is assumed positive. In Lemma 3.4(i) we do not require any of these assumptions (of course, the key point here is Lemma 2.3, which does not necessarily hold for a generic multidimensional continued fraction algorithm).

Theorem 3.5. *Let $p \in \Delta$. Then:*

(i) *p is rational iff $\mathrm{EPO}_M(p) = \{v_1\}$ iff p is a vertex of some $\mathcal{F}_t$;*
(ii) *p is dyadic iff $\mathrm{EPO}_T(p) = \{v_1\}$ iff p is a vertex of some $\mathcal{B}_t$.*

In particular, the set of rational points $\mathrm{GO}_M(v_1)$ is mapped bijectively by Φ to the set of dyadic points $\mathrm{GO}_T(v_1)$. Moreover, we have:

(iii) *$[\mathbb{Q}(p) : \mathbb{Q}] \leq n+1$ if p is M-preperiodic;*
(iv) *p is rational iff p is T-preperiodic.*

Proof. By construction, the M-counterimage of the set of vertices of $\mathcal{F}_t$ is the set of vertices of $\mathcal{F}_{t+1}$. Moreover, the vertices $v_1, \ldots, v_{n+1}$ of Δ are such that $M(v_1) = v_1$ and $M(v_j) = v_{j-1}$, for $2 \leq j \leq n+1$. Analogous statements hold for T, so in (i) and (ii) the equivalence of the second condition with the third is clear.

(i) If $\mathrm{EPO}_M(p) = \{v_1\}$, then $\mathbb{Q}(p) = \mathbb{Q}(v_1) = \mathbb{Q}$, and p is rational. Let p be rational, and let $l_1, \ldots, l_{n+1} \in \mathbb{Z}$ be its primitive projective coordinates. Let $p = \varphi(\bar{a})$. Then, for every $t \geq 0$, $p \in \Delta_{\bar{a}\restriction t}$ and, since $\Delta_{\bar{a}\restriction t}$ is unimodular, there exist $0 \leq k_1(t), \ldots, k_{n+1}(t) \in \mathbb{Z}$ such that

$$\begin{pmatrix} l_1 \\ \vdots \\ l_{n+1} \end{pmatrix} = V A_{a_0} \cdots A_{a_{t-1}} \begin{pmatrix} k_1(t) \\ \vdots \\ k_{n+1}(t) \end{pmatrix}.$$

Let $\big(c_1(t) \cdots c_{n+1}(t)\big)$ be the last row of $V A_{a_0} \cdots A_{a_{t-1}}$. The reader can easily prove (compare with [9, pp. 40-41]) that $1 \leq c_1(t) \leq \cdots \leq c_{n+1}(t)$ and that the sequence $\{c_2(t)\}_{t \geq 0}$ is nondecreasing, with limit ∞. Let t be such that $l_{n+1} < c_2(t)$. Then we must have $k_1(t) = 1$ and $k_2(t) = \cdots = k_{n+1}(t) = 0$. In other words, $(l_1 \cdots l_{n+1})^{tr}$ is the first column of $V A_{a_0} \cdots A_{a_{t-1}}$, and hence p is a vertex of $\mathcal{F}_t$.

(ii) If $\mathrm{EPO}_T(p) = \{v_1\}$, then $\mathbb{Z}[1/2](p) = \mathbb{Z}[1/2](v_1) = \mathbb{Z}[1/2]$, and p is dyadic. Conversely, let $p = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}[1/2]^n$ be dyadic, $p = \upsilon(\bar{a})$. Choose $m \geq 0$ such that $2^m p \in \mathbb{Z}^n$. Then, in projective coordinates, we have

$$T^{mn}(p) = T_{a_{mn-1}} \cdots T_{a_0} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \\ 1 \end{pmatrix}.$$

By Lemma 3.3, there exists $X \in \mathrm{GL}_{n+1}\,\mathbb{Z}$ such that

$$T^{mn}(p) = X T_0^{mn} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \\ 1 \end{pmatrix} = X \begin{pmatrix} 2^m & & & \\ & \ddots & & \\ & & 2^m & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \\ 1 \end{pmatrix} \in \mathbb{Z}^{n+1}.$$

Hence $T^{mn}(p)$ is one of the vertices of Δ and $T^{mn+n}(p) = v_1$.

(iii) Let $(\alpha_1, \ldots, \alpha_n) = q = M^s(q) = \varphi(\bar{a}) \in \mathrm{EPO}_M(p)$ for some $s > 0$. Then $(\alpha_1 \cdots \alpha_n\, 1)^{tr}$ is a right eigenvector for the matrix $M_{a_{s-1}} \cdots M_{a_0} \in \mathrm{GL}_{n+1}\,\mathbb{Z}$. The

corresponding eigenvalue ξ is a real algebraic unit of degree $\leq n+1$, and by Lemma 3.4(i) we have $\mathbb{Q}(p) = \mathbb{Q}(q) = \mathbb{Q}(\xi)$.

(iv) Let p be rational, and let $0 < k \in \mathbb{Z}$ be such that $kp \in \mathbb{Z}^n$. Since T_0 and T_1 have both integer entries, the forward T-orbit of p is contained in $\Delta \cap (k^{-1}\mathbb{Z})^n$, which is finite set; hence p is preperiodic. The reverse implication is analogous to (iii), using Lemma 3.4(ii). □

Finally, we discuss §1(3), i.e., the singularity of Φ w.r.t. the Lebesgue measure λ. We normalize λ so that $\lambda(\Delta) = 1$. Let $h(x_1, \ldots, x_n) \in L_1(\Delta, \lambda)$ be defined by

$$h(x_1, \ldots, x_n) = \frac{1}{x_1(x_1 - x_2 + 1)(x_1 - x_3 + 1) \cdots (x_1 - x_n + 1)},$$

and let μ be the probability measure on Δ induced by the density h, properly normalized (for the rest of this paper we are assuming $n \geq 2$, since otherwise μ is infinite):

$$\mu(A) = \int_A h\, d\lambda \Big/ \int_\Delta h\, d\lambda.$$

The Mönkemeyer map M preserves μ and is ergodic w.r.t. it [17, Theorems 23–24]. Note that in the above reference M appears as the restriction of the Selmer map to the absorbing n-simplex D in [17, Theorem 22], and the invariant density is $h'(x_1, \ldots, x_n) = \prod_i x_i^{-1}$. We leave to the reader —as a simple exercise in the calculus of Jacobians— to check that our h on Δ is the density corresponding to h' on D.

Given an n-simplex Λ in $\{x_{n+1} = 1\} \subset \mathbb{R}^{n+1}$, let L be an $(n+1) \times (n+1)$ real matrix whose columns express the vertices of Λ in projective coordinates, and such that the entries $L_{(n+1)1}, \ldots, L_{(n+1)(n+1)}$ in the last row are all > 0. Then one easily computes that

$$\lambda(\Lambda) = \frac{|\det(L)|}{L_{(n+1)1} \cdots L_{(n+1)(n+1)}}.$$

Applying this fact to $L = VB_{a_0} \cdots B_{a_{t-1}}$, we obtain

$$\lambda(\Gamma_{a_0 \ldots a_{t-1}}) = 2^{-t}. \tag{$*$}$$

Remember that if $\rho : X \to Y$ is a Borel map and σ is a Borel probability measure on X, then the *push-forward* of σ by ρ is the measure $\rho_* \sigma$ on Y defined by $(\rho_*\sigma)(A) = \sigma(\rho^{-1}A)$. If β denotes the Bernoulli measure on $\{0,1\}^{\mathbb{N}}$ obtained by giving 0 and 1 equal weight 1/2, the formula $(*)$ implies that $\upsilon_*\beta = \lambda$ (because such an identity holds on the simplexes $\Delta_{\bar{a}\restriction t}$, for $\bar{a} \in \{0,1\}^{\mathbb{N}}$ and $t \geq 0$, and these simplexes generate the Borel sets in Δ). Since υ induces a conjugation between the shift map S on $\{0,1\}^{\mathbb{N}}$ and the tent map T on Δ, it follows that T is ergodic w.r.t. λ, and hence M is ergodic w.r.t. $\Phi_*^{-1}\lambda$. Now, μ and $\Phi_*^{-1}\lambda$ are different (e.g., $(\Phi_*^{-1}\lambda)(\Delta_0) = 1/2 \neq \mu(\Delta_0)$), and are both ergodic w.r.t. the same transformation M. Therefore they are mutually singular [21, Theorem 6.10(iv)], and there exists a measurable set $A \subseteq \Delta$ such that $\mu(A) = 1$ and $\lambda\big(\Phi[A]\big) = 0$. Since $h \geq 1$ on Δ, we have $\mu \geq C\lambda$ for some constant $C > 0$. It follows that each of μ and λ is absolutely continuous w.r.t. the other, and in particular they have the same sets of full measure. We conclude that $\lambda(A) = 1$, and Φ is singular w.r.t. λ.

If $p = \varphi(\bar{a}) \in \Delta$, it is natural to look at the limit

$$\lim_{t \to \infty} \frac{\lambda\big(\Phi[\Delta_{\bar{a}\restriction t}]\big)}{\lambda(\Delta_{\bar{a}\restriction t})} \tag{$**$}$$

as an index of the singularity of Φ at p. As we already observed, $\lambda\big(\Phi[\Delta_{\bar{a}\restriction t}]\big) = 2^{-t}$. By the Shannon-McMillan-Breiman Theorem [3, §13] we have, for μ-all p (and hence for λ-all p, since μ and λ have the same nullsets), that

$$\lim_{t\to\infty} \frac{-\log\mu(\Delta_{\bar{a}\restriction t})}{t} = h_\mu, \tag{$***$}$$

where h_μ is the metrical entropy of M w.r.t. μ. Without loss of generality, we can assume that p is in the topological interior of Δ. For such a p, there exist t_0 and a constant $C > 0$ such that $C\mu(\Delta_{\bar{a}\restriction t}) \leq \lambda(\Delta_{\bar{a}\restriction t}) \leq C^{-1}\mu(\Delta_{\bar{a}\restriction t})$, for all $t \geq t_0$. It follows that in the identity $(***)$ we can safely substitute μ with λ. The value h_μ is explicitly computed in [1, §5.2] as follows: if

$$G(n) = \int_0^1 \frac{\big[\log(1+s)\big]^n}{s}\,ds,$$

then

$$h_\mu = \frac{(n+1)\,G(n)}{n\,G(n-1)}.$$

Taking logarithms in $(**)$ we have

$$\begin{aligned}\lim_{t\to\infty}\big[\log\lambda(\Gamma_{\bar{a}\restriction t}) - \log\lambda(\Delta_{\bar{a}\restriction t})\big] &= \lim_{t\to\infty}\left(-\log 2 - \frac{\log\lambda(\Delta_{\bar{a}\restriction t})}{t}\right)t \\ &= \lim_{t\to\infty} -(\log 2 - h_\mu)t.\end{aligned}$$

For $n = 2$ we have $h_\mu \sim 0.54807\ldots$ and, as shown in [1, §5.2], h_μ is monotonically increasing with n, tending to the limit $\log 2 \sim 0.69314\ldots$ —which is the topological entropy of M in every dimension— as n goes to infinity. We conclude that, for λ-all p and every $n \geq 2$, the limit $(**)$ approaches 0 exponentially fast. On the other hand, since $\lim_{n\to\infty}(\log 2 - h_\mu) = 0$, we might loosely say that the singularity of Φ decreases with the dimension.

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Department of Mathematics, University of Udine, via delle Scienze 208, 33100 Udine, Italy
E-mail address: `panti@dimi.uniud.it`